

Increasing-Chord Graphs On Point Sets*

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Abstract. We tackle the problem of constructing increasing-chord graphs spanning point sets. We prove that, for every point set P with n points, there exists an increasing-chord planar graph with $O(n)$ Steiner points spanning P . Further, we prove that, for every convex point set P with n points, there exists an increasing-chord graph with $O(n \log n)$ edges (and with no Steiner points) spanning P .

1 Introduction

A *proximity graph* is a geometric graph that can be constructed from a point set by connecting points that are “close”, for some local or global definition of proximity. Proximity graphs constitute a topic of research in which the areas of graph drawing and computational geometry nicely intersect. A typical graph drawing question in this topic asks to characterize the graphs that can be represented as a certain type of proximity graphs. A typical computational geometry question asks to design an algorithm to construct a proximity graph spanning a given point set.

Euclidean minimum spanning trees and Delaunay triangulations are famous examples of proximity graphs. Given a point set P , a *Euclidean minimum spanning tree* (MST) of P is a geometric tree with P as vertex set and with minimum total edge length; the *Delaunay triangulation* of P is a triangulation T such that no point in P lies inside the circumcircle of any triangle of T . From a computational geometry perspective, given a point set P with n points, an MST of P with maximum degree five exists [12] and can be constructed in $O(n \log n)$ time [4]; also, the Delaunay triangulation of P exists and can be constructed in $O(n \log n)$ time [4]. From a graph drawing perspective, every tree with maximum degree five admits a representation as an MST [12] and it is NP-hard to decide whether a tree with maximum degree six admits such a representation [7]; also, characterizing the class of graphs that can be represented as Delaunay triangulations is a deeply studied question, which still eludes a clear answer; see, e.g., [5,6]. Refer to the excellent survey by Liotta [10] for more on proximity graphs.

While proximity graphs have constituted a frequent topic of research in graph drawing and computational geometry, they gained a sudden peak in popularity even outside these communities in 2004, when Papadimitriou *et al.* [14] devised an elegant routing protocol that works effectively in all the networks that can be represented as a certain type of proximity graphs, called *greedy graphs*. For two points p and q in the plane,

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denote by \overline{pq} the straight-line segment having p and q as end-points, and by $|\overline{pq}|$ the length of \overline{pq} . A geometric path (v_1, \dots, v_n) is *greedy* if $|\overline{v_{i+1}v_n}| < |\overline{v_i v_n}|$, for every $1 \leq i \leq n-1$. A geometric graph G is *greedy* if, for every ordered pair of vertices u and v , there exists a greedy path from u to v in G . A result related to our paper is that, for every point set P , the Delaunay triangulation of P is a greedy graph [13].

In this paper we study *self-approaching* and *increasing-chord graphs*, that are types of proximity graphs defined by Alamdari *et al.* [2]. A geometric path $\mathcal{P} = (v_1, \dots, v_n)$ is *self-approaching* if, for every three points a, b , and c in this order on \mathcal{P} from v_1 to v_n (possibly a, b , and c are internal to segments of \mathcal{P}), it holds that $|\overline{bc}| < |\overline{ac}|$. A geometric graph G is *self-approaching* if, for every ordered pair of vertices u and v , G contains a self-approaching path from u to v ; also, G is *increasing-chord* if, for every pair of vertices u and v , G contains a path between u and v that is self-approaching both from u to v and from v to u ; thus, an increasing-chord graph is also self-approaching. The study of self-approaching and increasing-chord graphs is motivated by their relationship with greedy graphs (a self-approaching graph is also greedy), and by the fact that such graphs have a small geometric dilation, namely at most 5.3332 [9] (self-approaching graphs) and at most 2.094 [15] (increasing-chord graphs).

Alamdari *et al.* showed: (i) how to test in linear time whether a path in \mathbb{R}^2 is self-approaching; (ii) a characterization of the class of self-approaching trees; and (iii) how to construct, for every point set P with n points in \mathbb{R}^2 , an increasing-chord graph that spans P and uses $O(n)$ Steiner points.

In this paper we focus our attention on the problem of constructing increasing-chord graphs spanning given point sets in \mathbb{R}^2 . We prove two main results.

- We show that, for every point set P with n points, there exists an increasing-chord planar graph with $O(n)$ Steiner points spanning P . This answers a question of Alamdari *et al.* [2] and improves upon their result (iii) above, since our increasing-chord graphs are planar and contain increasing-chord paths between every pair of points, including the Steiner points (which is not the case for the graphs in [2]). It is interesting that our result is achieved by studying Gabriel triangulations, which are proximity graphs strongly related to Delaunay triangulations (a Gabriel triangulation of a point set P is a subgraph of the Delaunay triangulation of P). It has been proved in [2] that Delaunay triangulations are not, in general, self-approaching.
- We show that, for every convex point set P with n points, there exists an increasing-chord graph that spans P and that has $O(n \log n)$ edges (and no Steiner points).

2 Definitions and Preliminaries

A *geometric graph* (P, S) consists of a point set P in the plane and of a set S of straight-line segments (called *edges*) between points in P . A geometric graph is *planar* if no two of its edges cross. A planar geometric graph partitions the plane into connected regions called *faces*. The bounded faces are *internal* and the unbounded face is the *outer face*. A geometric planar graph is a *triangulation* if every internal face is delimited by a triangle and the outer face is delimited by a convex polygon.

Let p, q , and r be points in the plane. We denote by $\angle pqr$ the angle defined by a clockwise rotation around q bringing \overline{pq} to coincide with \overline{qr} .

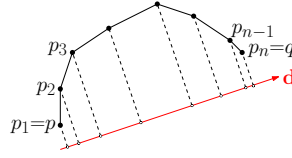


Fig. 1. A convex point set that is one-sided with respect to a directed straight line d .

A *convex combination* of a set of points $P = \{p_1, \dots, p_k\}$ is a point $\sum \alpha_i p_i$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for each $1 \leq i \leq k$. The *convex hull* \mathcal{H}_P of P is the set of points that can be expressed as a convex combination of the points in P . A *convex point set* P is such that no point is a convex combination of the others. Let P be a convex point set and d be a directed straight line not orthogonal to any line through two points of P . Order the points in P as their projections appear on d ; then, the *minimum point* and the *maximum point* of P with respect to d are the first and the last point in such an ordering. We say that P is *one-sided with respect to* d if the minimum and the maximum point of P with respect to d are consecutive along the border of \mathcal{H}_P . See Fig. 1. A *one-sided convex point set* is a convex point set that is one-sided with respect to some directed straight line d . The proof of our first lemma shows an algorithm to construct an increasing-chord planar graph spanning a one-sided convex point set.

Lemma 1. *Let P be any one-sided convex point set with n points. There exists an increasing-chord planar graph spanning P with $2n - 3$ edges.*

Proof: Assume that P is one-sided with respect to the positive x -axis x . Such a condition can be met after a suitable rotation of the Cartesian axes. Let $\{p_1, p_2, \dots, p_n\}$ be the points in P , ordered as their projections appear on x .

We show by induction on n that an increasing-chord planar graph G spanning P exists, in which all the edges on the border of \mathcal{H}_P are in G . If $n = 2$ then the graph with a single edge $\overline{p_1 p_2}$ is an increasing-chord planar graph spanning P . Next, assume that $n > 2$ and let p_j be a point with largest y -coordinate in P (possibly $j = 1$ or $j = n$). Point set $Q = P \setminus \{p_j\}$ is convex, one-sided with respect to x , and has $n - 1$ points. By induction, there exists an increasing-chord planar graph G' spanning Q in which all the edges on the border of \mathcal{H}_Q are in G' . Let G be the graph obtained by adding vertex p_j and edges $\overline{p_{j-1} p_j}$ and $\overline{p_j p_{j+1}}$ to G' . We have that G is planar, given that G' is planar and that edges $\overline{p_{j-1} p_j}$ and $\overline{p_j p_{j+1}}$ are on the border of \mathcal{H}_P . Further, all the edges on the border of \mathcal{H}_P are in G . Moreover, G contains an increasing-chord path between every pair of points in Q , by induction; also, G contains an increasing-chord path between p_j and every point p_i in Q , as one of the two paths on the border of \mathcal{H}_P connecting p_j and p_i is both x - and y -monotone, and hence increasing-chord by the results in [2]. Finally, G is a maximal outerplanar graph, hence it has $2n - 3$ edges. \square

The *Gabriel graph* of a point set P is the geometric graph that has an edge \overline{pq} between two points p and q if and only if the closed disk whose diameter is \overline{pq} contains no point of $P \setminus \{p, q\}$ in its interior or on its boundary. A *Gabriel triangulation* is a

triangulation that is the Gabriel graph of its point set P . We say that a point set P *admits* a Gabriel triangulation if the Gabriel graph of P is a triangulation. A triangulation is a Gabriel triangulation if and only if every angle of a triangle delimiting an internal face is acute [8]. See [8,10,11] for more properties about Gabriel graphs.

In Section 3 we will prove that every Gabriel triangulation is increasing-chord. A weaker version of the converse is also true, as proved in the following.

Lemma 2. *Let P be a set of points and let $G(P, S)$ be an increasing-chord graph spanning P . Then all the edges of the Gabriel graph of P are in S .*

Proof: Suppose, for a contradiction, that there exists an increasing-chord graph $G(P, S)$ and an edge \overline{uv} of the Gabriel graph of P such that $\overline{uv} \notin S$. Then, consider any increasing-chord path $\mathcal{P} = (u = w_1, w_2, \dots, w_k = v)$ in G . Since $\overline{uv} \notin S$, it follows that $k > 2$. Assume w.l.o.g. that w_1, w_2 , and w_k appear in this clockwise order on the boundary of triangle (w_1, w_2, w_k) . Since the closed disk with diameter \overline{uv} does not contain any point in its interior or on its boundary, it follows that $\angle w_k w_2 w_1 < 90^\circ$. If $\angle w_2 w_1 w_k \geq 90^\circ$, then $|w_1 w_k| < |w_2 w_k|$, a contradiction to the assumption that \mathcal{P} is increasing-chord. If $\angle w_2 w_1 w_k < 90^\circ$, then the altitude of triangle (w_1, w_2, w_k) incident to w_k hits $\overline{w_1 w_2}$ in a point h . Hence, $|h w_k| < |w_2 w_k|$, a contradiction to the assumption that \mathcal{P} is increasing-chord which proves the lemma. \square

3 Planar Increasing-Chord Graphs with Few Steiner Points

We show that, for any point set P , one can construct an increasing-chord planar graph $G(P', S)$ such that $P \subseteq P'$ and $|P'| \in O(|P|)$. Our result has two ingredients. The first one is that Gabriel triangulations are increasing-chord graphs. The second one is a result of Bern *et al.* [3] stating that, for any point set P , there exists a point set P' such that $P \subseteq P'$, $|P'| \in O(|P|)$, and P' admits a Gabriel triangulation. Combining these two facts proves our main result. The proof that Gabriel triangulations are increasing-chord graphs consists of two parts. In the first one, we prove that geometric graphs having a θ -path between every pair of points are increasing-chord. In the second one, we prove that in every Gabriel triangulation there exists a θ -path between every pair of points.

We introduce some definitions. The *slope* of a straight-line segment \overline{uv} is the angle spanned by a clockwise rotation around u that brings \overline{uv} to coincide with the positive x -axis. Thus, if θ is the slope of \overline{uv} , then $\theta + k \cdot 360^\circ$ is also the slope of \overline{uv} , $\forall k \in \mathbb{Z}$. A straight-line segment \overline{uv} is a θ -edge if its slope is in the interval $[\theta - 45^\circ; \theta + 45^\circ]$. Also, a geometric path $\mathcal{P} = (p_1, \dots, p_k)$ is a θ -path from p_1 to p_k if $\overline{p_i p_{i+1}}$ is a θ -edge, for every $1 \leq i \leq k - 1$. Consider a point a on a θ -path \mathcal{P} from p_1 to p_k . Then, the subpath \mathcal{P}_a of \mathcal{P} from a to p_k is also a θ -path. Moreover, denote by $W_\theta(a)$ the closed wedge with an angle of 90° incident to a and whose delimiting lines have slope $\theta - 45^\circ$ and $\theta + 45^\circ$; then \mathcal{P}_a is contained in $W_\theta(a)$ (see Fig. 2). We have the following:

Lemma 3. *Let \mathcal{P} be a θ -path from p_1 to p_k . Then, \mathcal{P} is increasing-chord.*

Proof: Lemma 3 in [9] states the following (see also [1]): A curve \mathcal{C} with endpoints p and q is self-approaching from p to q if and only if, for every point a on \mathcal{C} ,

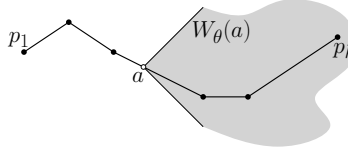


Fig. 2. Wedge $W_\theta(a)$ contains path \mathcal{P}_a .

there exists a closed wedge with an angle of 90° incident to a and containing the part of \mathcal{C} between a and q . By definition of θ -path, for every point a on \mathcal{P} , the closed wedge $W_\theta(a)$ with an angle of 90° incident to a and whose delimiting lines have slope $\theta - 45^\circ$ and $\theta + 45^\circ$ contains the subpath \mathcal{P}_a of \mathcal{P} from a to p_k . Hence, by Lemma 3 in [9], \mathcal{P} is self-approaching from p_1 to p_k . An analogous proof shows that \mathcal{P} is self-approaching from p_k to p_1 , given that \mathcal{P} is a $(\theta + 180^\circ)$ -path from p_k to p_1 . \square

We now prove that Gabriel triangulations contain θ -paths.

Lemma 4. *Let G be a Gabriel triangulation on a point set P . For every two points $s, t \in P$, there exists an angle θ such that G contains a θ -path from s to t .*

Proof: Consider any two points $s, t \in P$. Clockwise rotate G of an angle ϕ so that $y(s) = y(t)$ and $x(s) < x(t)$. Observe that, if there exists a θ -path from s to t after the rotation, then there exists a $(\theta + \phi)$ -path from s to t before the rotation.

A θ -path (p_1, \dots, p_k) in G is *maximal* if there is no $z \in P$ such that $\overline{p_k z}$ is a θ -edge. For every maximal θ -path $\mathcal{P} = (p_1, \dots, p_k)$ in G , p_k lies on the border of \mathcal{H}_P . Namely, assume the converse, for a contradiction. Since G is a Gabriel triangulation, the angle between any two consecutive edges incident to an internal vertex of G is smaller than 90° , thus there is a θ -edge incident to p_k . This contradicts the maximality of \mathcal{P} . A maximal θ -path $(s = p_1, \dots, p_k)$ is *high* if either (a) $y(p_k) > y(t)$ and $x(p_k) < x(t)$, or (b) $\overline{p_i p_{i+1}}$ intersects the vertical line through t at a point above t , for some $1 \leq i \leq k - 1$. Symmetrically, a maximal θ -path $(s = p_1, \dots, p_k)$ is *low* if either (a) $y(p_k) < y(t)$ and $x(p_k) < x(t)$, or (b) $\overline{p_i p_{i+1}}$ intersects the vertical line through t at a point below t , for some $1 \leq i \leq k - 1$. High and low $(\theta + 180^\circ)$ -paths starting at t can be defined analogously. The proof of the lemma consists of two main claims.

Claim 1. If a maximal θ -path \mathcal{P}_s starting at s and a maximal $(\theta + 180^\circ)$ -path \mathcal{P}_t starting at t exist such that \mathcal{P}_s and \mathcal{P}_t are both high or both low, for some $-45^\circ \leq \theta \leq 45^\circ$, then there exists a θ -path in G from s to t .

Claim 2. For some $-45^\circ \leq \theta \leq 45^\circ$, there exist a maximal θ -path \mathcal{P}_s starting at s and a maximal $(\theta + 180^\circ)$ -path \mathcal{P}_t starting at t that are both high or both low.

Observe that Claims 1 and 2 imply the lemma.

We now prove Claim 1. Suppose that G contains a maximal high θ -path \mathcal{P}_s starting at s and a maximal high $(\theta + 180^\circ)$ -path \mathcal{P}_t starting at t , for some $-45^\circ \leq \theta \leq 45^\circ$. If \mathcal{P}_s and \mathcal{P}_t share a vertex $v \in P$, then the subpath of \mathcal{P}_s from s to v and the subpath of \mathcal{P}_t from v to t form a θ -path in G from s to t . Thus, it suffices to show that \mathcal{P}_s and \mathcal{P}_t share a vertex. For a contradiction assume the converse. Let p_s and p_t be the

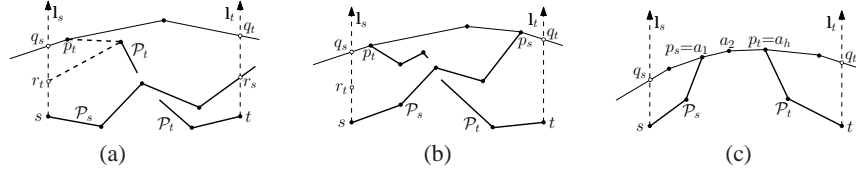


Fig. 3. Paths \mathcal{P}_s and \mathcal{P}_t intersect if: (a) $x(p_s) \geq x(t)$, (b) $x(s) < x(p_t) < x(p_s) < x(t)$, and (c) $x(s) < x(p_s) < x(p_t) < x(t)$.

end-vertices of \mathcal{P}_s and \mathcal{P}_t different from s and t , respectively. Recall that p_s and p_t lie on the border of \mathcal{H}_P . Denote by l_s and l_t the vertical half-lines starting at s and t , respectively, and directed towards increasing y -coordinates; also, denote by q_s and q_t the intersection points of l_s and l_t with the border of \mathcal{H}_P , respectively. Finally, denote by Q the curve obtained by clockwise following the border of \mathcal{H}_P from q_s to q_t .

Assume that $x(p_s) \geq x(t)$, as in Fig. 3(a). Path \mathcal{P}_s starts at s and passes through a point r_s on l_t (possibly $r_s = q_t$), given that $x(p_s) \geq x(t)$. Path \mathcal{P}_t starts at t and either passes through a point r_t on l_s , or ends at a point p_t on Q , depending on whether $x(p_t) \leq x(s)$ or $x(p_t) > x(s)$, respectively. Since \mathcal{P}_s is x -monotone and lies in \mathcal{H}_P , it follows that r_t and p_t are above or on \mathcal{P}_s ; also, t is below \mathcal{P}_s given that \mathcal{P}_s is a high path. It follows \mathcal{P}_s and \mathcal{P}_t intersect, hence they share a vertex given that G is planar.

Analogously, if $x(p_t) \leq x(s)$, then \mathcal{P}_s and \mathcal{P}_t share a vertex.

If $x(p_t) = x(p_s)$, then $\mathcal{P}_s \cup \mathcal{P}_t$ is a θ -path from s to t .

Next, if $x(s) < x(p_t) < x(p_s) < x(t)$, as in Fig. 3(b), then the end-points of \mathcal{P}_s and \mathcal{P}_t alternate along the boundary of the region R that is the intersection of \mathcal{H}_P , of the half-plane to the right of l_s , and of the half-plane to the left of l_t . Since \mathcal{P}_s and \mathcal{P}_t are x -monotone, they lie in R , thus they intersect, and hence they share a vertex.

Finally, assume that $x(s) < x(p_s) < x(p_t) < x(t)$, as in Fig. 3(c). Let a_1, \dots, a_h be the clockwise order of the points along Q , starting at $p_s = a_1$ and ending at $a_h = p_t$. By the assumption $x(p_s) < x(p_t)$ we have $h \geq 2$. We prove that $\overline{a_1 a_2}$ is a θ -edge. Suppose, for a contradiction, that $\overline{a_1 a_2}$ is not a θ -edge. Since the slope of $\overline{a_1 a_2}$ is larger than -90° and smaller than 90° , it is either larger than $\theta + 45^\circ$ and smaller than 90° , or it is larger than -90° and smaller than $\theta - 45^\circ$. First, assume that the slope of $\overline{a_1 a_2}$ is larger than $\theta + 45^\circ$ and smaller than 90° , as in Fig. 4(a). Since the slope of $\overline{s a_1}$ is between $\theta - 45^\circ$ and $\theta + 45^\circ$, it follows that a_1 is below the line composed of $\overline{s a_2}$ and $\overline{a_2 t}$, which contradicts the assumption that a_1 is on Q . Second, if the slope of $\overline{a_1 a_2}$ is larger than -90° and smaller than $\theta - 45^\circ$, then we distinguish two further cases. In the first case, represented in Fig. 4(b), the slope of $\overline{a_1 t}$ is larger than $\theta - 45^\circ$, hence a_2 is below the line composed of $\overline{s a_1}$ and $\overline{a_1 t}$, which contradicts the assumption that a_2 is on Q . In the second case, represented in Fig. 4(c), the slope of $\overline{a_1 t}$ is in the interval $[-90^\circ; \theta - 45^\circ]$. It follows that the slope of $\overline{t a_1}$ is in the interval $[90^\circ; \theta + 135^\circ]$; since the slope of $\overline{t a_h}$ is smaller than the one of $\overline{t a_1}$, we have that \mathcal{P}_t is not a $(\theta + 180^\circ)$ -path. This contradiction proves that $\overline{a_1 a_2}$ is a θ -edge. However, this contradicts the assumption that \mathcal{P}_s is a maximal θ -path, and hence concludes the proof of Claim 1.

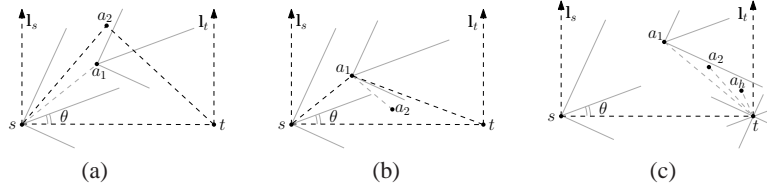


Fig. 4. Illustration for the proof that $\overline{a_1 a_2}$ is a θ -edge.

We now prove Claim 2. First, we prove that, for *every* θ in the interval $[-45^\circ; 45^\circ]$, there exists a maximal θ -path starting at s that is low or high. Indeed, it suffices to prove that there exists a θ -edge incident to s , as such an edge is also a θ -path starting at s , and the existence of a θ -path starting at s implies the existence of a maximal θ -path starting at s . Consider a straight-line segment e_θ that is the intersection of a directed half-line incident to s with slope θ and of a disk of arbitrarily small radius centered at s . If e_θ is internal to \mathcal{H}_P , then consider the two edges e_1 and e_2 of G that are encountered when counter-clockwise and clockwise rotating e_θ around s , respectively. Then, e_1 or e_2 is a θ -edge, as the angle spanned by a clockwise rotation bringing e_1 to coincide with e_2 is smaller than 90° , given that G is a Gabriel triangulation, and e_θ is encountered during such a rotation. If e_θ is outside \mathcal{H}_P , which might happen if s on the boundary of \mathcal{H}_P , then assume that the slope of e_θ is in the interval $[0^\circ; 45^\circ]$ (the case in which the slope of e_θ is in the interval $[-45^\circ; 0^\circ]$ is analogous). Then, the angle spanned by a clockwise rotation bringing e_θ to coincide with \overline{st} is at most 45° . Since \overline{st} is in interior or on the boundary of \mathcal{H}_P , an edge e_1 of G is encountered during such a rotation, hence e_1 is a θ -edge. An analogous proof shows that, for *every* θ in the interval $[-45^\circ; 45^\circ]$, there exists a maximal $(\theta + 180^\circ)$ -path starting at t that is low or high.

Second, we prove that, for *some* $\theta \in [-45^\circ; 45^\circ]$, there exist a maximal low θ -path and a maximal high θ -path both starting at s . All the maximal (-45°) -paths (all the maximal (45°) -paths) starting at s are low (resp. high), given that every edge on these paths has slope in the interval $[-90^\circ; 0^\circ]$ (resp. $[0^\circ; 90^\circ]$). Thus, let θ be the smallest constant in the interval $[-45^\circ; 45^\circ]$ such that a maximal high θ -path exists. We prove that there also exists a maximal low θ -path starting at s . Consider an arbitrarily small $\epsilon > 0$. By assumption, there exists no high $(\theta - \epsilon)$ -path. Hence, from the previous argument there exists a low $(\theta - \epsilon)$ -path \mathcal{P} . If ϵ is sufficiently small, then no edge of \mathcal{P} has slope in the interval $[\theta - 45^\circ - \epsilon; \theta - 45^\circ]$. Thus every edge of \mathcal{P} has slope in the interval $[\theta - 45^\circ; \theta + 45^\circ - \epsilon]$, hence \mathcal{P} is a maximal low θ -path starting at s .

Since there exist a maximal high θ -path starting at s , a maximal low θ -path starting at s , and a maximal $(\theta + 180^\circ)$ -path starting at t that is low or high, it follows that there exist a maximal θ -path \mathcal{P}_s starting at s and a maximal $(\theta + 180^\circ)$ -path \mathcal{P}_t starting at t that are both high or both low. This proves Claim 2 and hence the lemma. \square

Lemma 3 and Lemma 4 immediately imply the following.

Corollary 1. *Any Gabriel triangulation is increasing-chord.*

We are now ready to state the main result of this section.

Theorem 1. *Let P be a point set with n points. One can construct in $O(n \log n)$ time an increasing-chord planar graph $G(P', S)$ such that $P \subseteq P'$ and $|P'| \in O(n)$.*

Proof: Bern, Eppstein, and Gilbert [3] proved that, for any point set P , there exists a point set P' with $P \subseteq P'$ and $|P'| \in O(n)$ such that P' admits a Gabriel triangulation G . Both P' and G can be computed in $O(n \log n)$ time [3]. By Corollary 1, G is increasing-chord, which concludes the proof. \square

We remark that $o(|P|)$ Steiner points are not always enough to augment a point set P to a point set that admits a Gabriel triangulation. Namely, consider any point set B with $O(1)$ points that admits no Gabriel triangulation. Construct a point set P out of $|P|/|B|$ copies of B placed “far apart” from each other, so that any triangle with two points in different copies of B is obtuse. Then, a Steiner point has to be added inside the convex hull of each copy of B to obtain a point set that admits a Gabriel triangulation.

4 Increasing-Chord Convex Graphs with Few Edges

In this section we prove the following theorem;

Theorem 2. *For every convex point set P with n points, there exists an increasing-chord geometric graph $G(P, S)$ such that $|S| \in O(n \log n)$.*

The main idea behind the proof of Theorem 2 is that any convex point set P can be decomposed into some one-sided convex point sets P_1, \dots, P_k (which by Lemma 1 admit increasing-chord spanning graphs with linearly many edges) in such a way that every two points of P are part of some P_i and that $\sum |P_i|$ is small. In order to perform such a decomposition, we introduce the concept of *balanced $(\mathbf{d}_1, \mathbf{d}_2)$ -partition*.

Let P be a convex point set and let \mathbf{d} be a directed straight line not orthogonal to any line through two points of P . See Fig. 5. Let $p_a(\mathbf{d})$ and $p_b(\mathbf{d})$ be the minimum and maximum point of P with respect to \mathbf{d} , respectively. Let $P_1(\mathbf{d})$ be composed of those points in P that are encountered when clockwise walking along the boundary of \mathcal{H}_P from $p_a(\mathbf{d})$ to $p_b(\mathbf{d})$, where $p_a(\mathbf{d}) \in P_1(\mathbf{d})$ and $p_b(\mathbf{d}) \notin P_1(\mathbf{d})$. Analogously, let $P_2(\mathbf{d})$ be composed of those points in P that are encountered when clockwise walking along the boundary of \mathcal{H}_P from $p_b(\mathbf{d})$ to $p_a(\mathbf{d})$, where $p_b(\mathbf{d}) \in P_2(\mathbf{d})$ and $p_a(\mathbf{d}) \notin P_2(\mathbf{d})$.

Let \mathbf{d}_1 and \mathbf{d}_2 be two directed straight lines not orthogonal to any line through two points of P , where the clockwise rotation that brings \mathbf{d}_1 to coincide with \mathbf{d}_2 is at most 180° . The $(\mathbf{d}_1, \mathbf{d}_2)$ -partition of P partitions P into subsets $P_a = P_1(\mathbf{d}_1) \cap P_1(\mathbf{d}_2)$, $P_b = P_1(\mathbf{d}_1) \cap P_2(\mathbf{d}_2)$, $P_c = P_2(\mathbf{d}_1) \cap P_1(\mathbf{d}_2)$, and $P_d = P_2(\mathbf{d}_1) \cap P_2(\mathbf{d}_2)$. Note that every point in P is contained in one of P_a, P_b, P_c , and P_d . A $(\mathbf{d}_1, \mathbf{d}_2)$ -partition of P is *balanced* if $|P_a| + |P_d| \leq \frac{|P|}{2} + 1$ and $|P_b| + |P_c| \leq \frac{|P|}{2} + 1$. We now argue that, for every point set P , a balanced $(\mathbf{d}_1, \mathbf{d}_2)$ -partition of P always exists, even if \mathbf{d}_1 is arbitrarily prescribed.

Lemma 5. *Let P be a convex point set and let \mathbf{d}_1 be a directed straight line not orthogonal to any line through two points of P . Then, there exists a directed straight line \mathbf{d}_2 that is not orthogonal to any line through two points of P such that the $(\mathbf{d}_1, \mathbf{d}_2)$ -partition of P is balanced.*

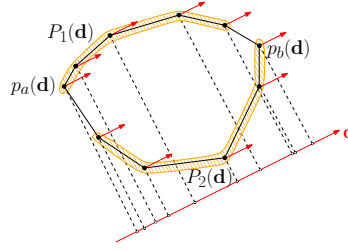


Fig. 5. Subsets $P_1(\mathbf{d})$ and $P_2(\mathbf{d})$ of a point set P determined by a directed straight line \mathbf{d} .

Proof: Denote by $q_1 = p_a(\mathbf{d}_1), q_2, \dots, q_l, q_{l+1} = p_b(\mathbf{d}_1)$ the points of P encountered when clockwise walking on the boundary of \mathcal{H}_P from $p_a(\mathbf{d}_1)$ to $p_b(\mathbf{d}_1)$. Also, denote by $r_1 = p_b(\mathbf{d}_1), r_2, \dots, r_m, r_{m+1} = p_a(\mathbf{d}_1)$ the points of P encountered when clockwise walking on the boundary of \mathcal{H}_P from $p_b(\mathbf{d}_1)$ to $p_a(\mathbf{d}_1)$.

Initialize \mathbf{d}_2 to be a directed straight line coincident with \mathbf{d}_1 . When $\mathbf{d}_2 = \mathbf{d}_1$, we have $P_a = \{q_1, q_2, \dots, q_l\}$, $P_d = \{r_1, r_2, \dots, r_m\}$, $P_b = \emptyset$, and $P_c = \emptyset$. We now clockwise rotate \mathbf{d}_2 until it is opposite to \mathbf{d}_1 (that is, parallel and pointing in the opposite direction). As we rotate \mathbf{d}_2 , sets $P_1(\mathbf{d}_2)$ and $P_2(\mathbf{d}_2)$ change, hence sets P_a , P_b , P_c , and P_d change as well. When \mathbf{d}_2 is opposite to \mathbf{d}_1 , we have $P_a = \emptyset$, $P_d = \emptyset$, $P_b = \{q_1, q_2, \dots, q_l\}$, and $P_c = \{r_1, r_2, \dots, r_m\}$. We will argue that there is a moment during such a rotation of \mathbf{d}_2 in which the corresponding $(\mathbf{d}_1, \mathbf{d}_2)$ -partition of P is balanced. Assume that at any time instant during the rotation of \mathbf{d}_2 the following hold (see Figs. 6(a)–(b)):

- $P_b = \{q_1, q_2, \dots, q_j\}$ (possibly P_b is empty);
- $P_a = \{q_{j+1}, q_{j+2}, \dots, q_l\}$ (possibly P_a is empty);
- $P_c = \{r_1, r_2, \dots, r_k\}$ (possibly P_c is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \dots, r_m\}$ (possibly P_d is empty); and
- q_{j+1} and r_{k+1} are the minimum and maximum point of P w.r.t. \mathbf{d}_2 , respectively.

The assumption is indeed true when \mathbf{d}_2 starts moving, with $j = 0$ and $k = 0$.

As we keep on clockwise rotating \mathbf{d}_2 , at a certain moment \mathbf{d}_2 becomes orthogonal to $\overline{q_{j+1}q_{j+2}}$ or to $\overline{r_{k+1}r_{k+2}}$ (or to both if $\overline{q_{j+1}q_{j+2}}$ and $\overline{r_{k+1}r_{k+2}}$ are parallel). Thus, as we keep on clockwise rotating \mathbf{d}_2 , sets P_a , P_b , P_c , and P_d change. Namely:

If \mathbf{d}_2 becomes orthogonal first to $\overline{q_{j+1}q_{j+2}}$ and then to $\overline{r_{k+1}r_{k+2}}$, then as \mathbf{d}_2 rotates clockwise after the position in which it is orthogonal to $\overline{q_{j+1}q_{j+2}}$, we have

- $P_b = \{q_1, q_2, \dots, q_j, q_{j+1}\}$;
- $P_a = \{q_{j+2}, q_{j+3}, \dots, q_l\}$ (possibly P_a is empty);
- $P_c = \{r_1, r_2, \dots, r_k\}$ (possibly P_c is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \dots, r_m\}$ (possibly P_d is empty); and
- q_{j+2} and r_{k+1} are the minimum and maximum point of P w.r.t. \mathbf{d}_2 , respectively.

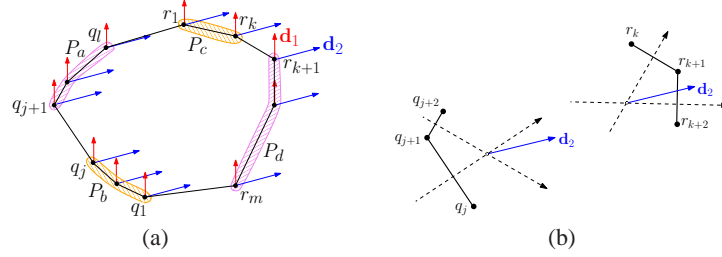


Fig. 6. (a) Sets P_a , P_b , P_c , and P_d at a certain time instant during the rotation of d_2 . (b) The slope of d_2 with respect to the slopes of the lines orthogonal to $\overline{q_j q_{j+1}}$, to $\overline{q_{j+1} q_{j+2}}$, to $\overline{r_k r_{k+1}}$, and to $\overline{r_{k+1} r_{k+2}}$.

If d_2 becomes orthogonal first to $\overline{r_{k+1} r_{k+2}}$ and then to $\overline{q_{j+1} q_{j+2}}$, then as d_2 rotates clockwise after the position in which it is orthogonal to $\overline{r_{k+1} r_{k+2}}$, we have that P_a and P_b stay unchanged, that r_{k+1} passes from P_d to P_c , and that q_{j+1} and r_{k+2} are the minimum and maximum point of P w.r.t. d_2 , respectively.

If d_2 becomes orthogonal to $\overline{q_{j+1} q_{j+2}}$ and $\overline{r_{k+1} r_{k+2}}$ simultaneously, then as d_2 rotates clockwise after the position in which it is orthogonal to $\overline{q_{j+1} q_{j+2}}$, we have that q_{j+1} passes from P_a to P_b , that r_{k+1} passes from P_d to P_c , and that q_{j+2} and r_{k+2} are the minimum and maximum point of P w.r.t. d_2 , respectively.

Observe that:

1. whenever sets P_a , P_b , P_c , and P_d change, we have that $|P_a| + |P_d|$ and $|P_b| + |P_c|$ change at most by two;
2. when d_2 starts rotating we have that $|P_a| + |P_d| = |P|$, and when d_2 stops rotating we have that $|P_a| + |P_d| = 0$;
3. when d_2 starts rotating we have that $|P_b| + |P_c| = 0$, and when d_2 stops rotating we have that $|P_b| + |P_c| = |P|$; and
4. $|P_a| + |P_b| + |P_c| + |P_d| = |P|$ holds at any time instant.

By continuity, there is a time instant in which $|P_a| + |P_d| = \lfloor |P|/2 \rfloor$ and $|P_b| + |P_c| = \lceil |P|/2 \rceil$, or in which $|P_a| + |P_d| = \lfloor |P|/2 \rfloor + 1$ and $|P_b| + |P_c| = \lceil |P|/2 \rceil - 1$. This completes the proof of the lemma. \square

We now show how to use Lemma 5 in order to prove Theorem 2.

Let P be any point set. Assume that no two points of P have the same y -coordinate. Such a condition is easily met after rotating the Cartesian axes. Denote by l a vertical straight line directed towards increasing y -coordinates. Each of $P_1(l)$ and $P_2(l)$ is convex and one-sided with respect to l . By Lemma 1, there exist increasing-chord graphs $G_1 = (P_1(l), S_1)$ and $G_2 = (P_2(l), S_2)$ with $|S_1| < 2|P_1(l)|$ and $|S_2| < 2|P_2(l)|$. Then, graph $G(P, S_1 \cup S_2)$ has less than $2(|P_1(l)| + |P_2(l)|) = 2|P|$ edges and contains an increasing-chord path between every pair of vertices in $P_1(l)$ and between every pair of vertices in $P_2(l)$. However, G does not have increasing-chord paths between any pair (a, b) of vertices such that $a \in P_1(l)$ and $b \in P_2(l)$.

We now present and prove the following claim. Consider a convex point set Q and a directed straight line d_1 not orthogonal to any line through two points of Q . Then, there exists a geometric graph $H(Q, R)$ that contains an increasing-chord path between every point in $Q_1(d_1)$ and every point in $Q_2(d_1)$, such that $|R| \in O(|Q| \log |Q|)$.

The application of the claim with $Q = P$ and $d_1 = l$ provides a graph $H(P, R)$ that contains an increasing-chord path between every pair (a, b) of vertices such that $a \in P_1(l)$ and $b \in P_2(l)$. Thus, the union of G and H is an increasing-chord graph with $O(|P| \log |P|)$ edges spanning P . Therefore, the above claim implies Theorem 2.

We show an inductive algorithm to construct H . Let $f(Q, d_1)$ be the number of edges that H has as a result of the application of our algorithm on a point set Q and a directed straight-line d_1 . Also, let $f(n) = \max\{f(Q, d_1)\}$, where the maximum is among all point sets Q with $n = |Q|$ points and among all the directed straight-lines d_1 that are not orthogonal to any line through two points of Q .

Let Q be any convex point set with n points and let d_1 be any directed straight line not orthogonal to any line through two points of Q . By Lemma 5, there exists a directed straight line not orthogonal to any line through two points of Q and such that the (d_1, d_2) -partition of Q is balanced.

Let $Q_a = Q_1(d_1) \cap Q_1(d_2)$, let $Q_b = Q_1(d_1) \cap Q_2(d_2)$, let $Q_c = Q_2(d_1) \cap Q_1(d_2)$, and let $Q_d = Q_2(d_1) \cap Q_2(d_2)$.

Point set $Q_a \cup Q_c$ is convex and one-sided with respect to d_2 . By Lemma 1 there exists an increasing-chord graph $H_1(Q_a \cup Q_c, R_1)$ with $|R_1| < 2(|Q_a| + |Q_c|)$ edges. Analogously, by Lemma 1 there exists an increasing-chord graph $H_2(Q_b \cup Q_d, R_2)$ with $|R_2| < 2(|Q_b| + |Q_d|)$ edges.

Hence, there exists a graph $H_3(Q, R_1 \cup R_2)$ with $|R_1 \cup R_2| < 2(|Q_a| + |Q_c| + |Q_b| + |Q_d|) = 2|Q| = 2n$ edges containing an increasing-chord path between every point in Q_a and every point in Q_c , and between every point in Q_b and every point in Q_d . However, G does not have an increasing-chord path between any point in Q_a and any point in Q_d , and does not have an increasing-chord path between any point in Q_b and any point in Q_c .

By Lemma 5, it holds that $|Q_a| + |Q_d| \leq \frac{n}{2} + 1$ and $|Q_b| + |Q_c| \leq \frac{n}{2} + 1$. By definition, we have $f(Q_a \cup Q_d, d_1) \leq f(|Q_a| + |Q_d|) \leq f(\frac{n}{2} + 1)$. Analogously, it holds that $f(Q_b \cup Q_c, d_1) \leq f(|Q_b| + |Q_c|) \leq f(\frac{n}{2} + 1)$. Hence, $f(n) \leq 2n + 2f(\frac{n}{2} + 1) \in O(n \log n)$. This proves the claim and hence Theorem 2.

5 Conclusions

We considered the problem of constructing increasing-chord graphs spanning point sets. We proved that, for every point set P , there exists a planar increasing-chord graph $G(P', S)$ with $P \subseteq P'$ and $|P'| \in O(|P|)$. We also proved that, for every convex point set P , there exists an increasing-chord graph $G(P, S)$ with $|S| \in O(|P| \log |P|)$.

Despite our research efforts, the main question on this topic remains open:

Problem 1. Is it true that, for every (convex) point set P , there exists an increasing-chord planar graph $G(P, S)$?

One of the directions we took in order to tackle this problem is to assume that the points in P lie on a constant number of straight lines. While a simple modification of the proof of Lemma 1 allows us to prove that an increasing-chord planar graph always exists spanning a set of points lying on two straight lines, it is surprising and disheartening that we could not prove a similar result for sets of points lying on three straight lines. The main difficulty seems to lie in the construction of planar increasing-chord graphs spanning sets of points lying on the boundary of an acute triangle.

Gabriel graphs naturally generalize to higher dimensions, where empty balls replace empty disks. In Section 3 we showed that, for points in \mathbb{R}^2 , every Gabriel triangulation is increasing-chord. Can this result be generalized to higher dimensions?

Problem 2. Is it true that, for every point set P in \mathbb{R}^d , any Gabriel triangulation of P is increasing-chord?

Finally, it would be interesting to understand if increasing-chord graphs with few edges can be constructed for any (possibly non-convex) point set:

Problem 3. Is it true that, for every point set P , there exists an increasing-chord graph $G(P, S)$ with $|S| \in o(|P|^2)$?

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